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ON A LINEAR GUIDANCE GAME PROBLEM

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A guidance game problem is analyzed for a linear conflict-controlled system when the game's payoff has the meaning of the Euclidean distance of the phase point from the origin. A certain modification is suggested for the extremal aiming rule [1], which under specific conditions quarantees one of the players a result not worse than in the corresponding program problem on maximin for the initial position. The paper relies on the idea of a position differential game, developed in [1, 2].

1. We consider a conflict-controlled system described by the vector differential equation

$$y^{\cdot} = A(t)y + B(t)u - C(t)v, \quad u \in P, \quad v \in Q$$

where y is the n-dimensional phase vector, u and v are r-dimensional controls of the first and second players, respectively, $A(t)$, $B(t)$, and $C(t)$ are matrices of appropriate dimensions, continuous in t and P and Q are convex closed bounded sets. The game is analyzed on a specified interval $t_0 \leqslant t \leqslant \vartheta$ and the payoff $\gamma[\vartheta]$ is represented by the equality

$$\gamma[\vartheta] = \| \{y[\vartheta]\}_m \|$$

Here and subsequently $\| x \|$ is the Euclidean norm of vector x and $\{x\}_m$ is a vector composed of the first m components of vector x. The system being analyzed can be reduced by a nonsingular linear transformation to the form (see [2])

$$x^{\cdot} = B(t)u - C(t)v, \quad u \in P, \quad v \in Q \tag{1.1}$$

where x is an m-dimensional vector, $B(t)$ and $C(t)$ are matrices continuous in t and the game's payoff has the form

$$\gamma[\vartheta] = \| x[\vartheta] \| \tag{1.2}$$

In what follows it is convenient to use a system transformed to form (1.1).

The first player chooses a control $u[t] \in P$ and tries to minimize the quantity $\gamma[\vartheta]$ on the trajectories $x[t]$ $(t_0 \leqslant t \leqslant \vartheta,\ x[t_0] = x_0)$ of system (1.1), realized under his control $u[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ in pair with any integrable realization $v[t] \in Q$ of the second player's control. The second player has the opposing purpose and tries to maximize the quantity $\gamma[\vartheta]$ in (1.2).

The admissible strategies **U** and **V** of the first and second players, respectively, are specified to be convex, closed and upper semi-continuous by inclusion under a change of position by sets $U(t, x) \subset P$ and $V(t, x) \subset Q$; by motions we mean the solutions of the corresponding contingent equations. Let $(\gamma[\vartheta] \mid t_0, x_0, u, v)$ be a realization of the quantity $\gamma[\vartheta]$ (1.2), corresponding to the initial position $\{t_0, x_0\}$ under certain controls u and v.

P r o b l e m 1. Among the first player's admissible strategies **U** find the strategy **U*** which for any initial position guarantees the game result

$$(\gamma[\vartheta] \mid t_0, x_0, \mathbf{U}^*, v) \leqslant \varepsilon_0(t_0, x_0)$$

under any admissible control method of the second player.

P r o b l e m 2. Among the second player's admissible strategies **V** find the strategy **V*** which for any initial position $\{t_0, x_0\}$ guarantees the game result

$$(\gamma[\vartheta] \mid t_0, x_0, u, \mathbf{V}^*) \geqslant \varepsilon_0(t_0, x_0)$$

under any admissible control method of the first player.

In these problems the quantity $\varepsilon_0(t_0, x_0)$ is the program maximin for the initial position $\{t_0, x_0\}$ and is defined by the equality [1]

$$\varepsilon_0(t_0, x_0) = \max_{\|l\|=1} \Big[\int_{t_0}^{\vartheta} \max_{v \in Q} l'C(t)v(t)\,dt - \int_{t_0}^{\vartheta} \max_{u \in P} l'B(t)u(t)\,dt - l'x_0\Big] \tag{1.3}$$

if the right-hand side of this equality is positive; otherwise, $\varepsilon_0(t_0, x_0) = 0$. The prime denotes transposition. We assume that $\varepsilon_0(t_0, x_0) > 0$ for the initial position $\{t_0, x_0\}$.

2. Let the following condition [2] be fulfilled: the function

$$\varkappa(l, t) = \max_{u \in P} l'B(t)u - \max_{v \in Q} l'C(t)v \tag{2.1}$$

is convex in l for all $t \in [t_0, \vartheta]$ (Condition A). This is a necessary and sufficient condition for the maximum on the right-hand side of (1.3) to be achieved on a unique vector $l_0 = l_0(t_0, x_0)$. In addition, when this condition is fulfilled the function $\varkappa(l, t)$ is [2, 3] the support function of the convex closed set

$$H(t) = \bigcap_{v \in Q} \{B(t)P - C(t)v\} \tag{2.2}$$

We shall examine the program controls $u^0(t,\ l_0)$ and $v^0(t,\ l_0)$, $t_0 \leqslant t \leqslant \vartheta$, satisfying for almost all t the maximum conditions

$$l_0'B\ (t)\ u^\circ(t,\ l_0) = \max_{u \in P} l_0'B\ (t)u \tag{2.3}$$

$$l_0'C\ (t)v^\circ(t,\ l_0) = \max_{v \in Q} l_0'C\ (t)v \tag{2.4}$$

where l_0 is that vector $l_0 = l_0\ (t_0,\ x_0)$ on which the maximum on the right-hand side of (1.3) is achieved.

L e m m a 1. If sets P and Q are convex and Condition A is valid, then program controls $u^\circ\ (t,\ l_0)$ and $v^\circ\ (t,\ l_0)$, measurable in t, exist and satisfy maximum conditions (2.3) and (2.4) for almost all $t \in [t_0,\ \vartheta]$, for which the inclusion

$$h^\circ\ (t,\ l_0) = \{B\ (t)\ u^\circ\ (t,\ l_0) - C\ (t)\ v^\circ\ (t,\ l_0)\} \in H\ (t) \tag{2.5}$$

holds almost everywhere on the interval $[t_0,\ \vartheta]$.

P r o o f. The functions $\max_{u \in P} l'B\ (t)\ u$ and $\max_{v \in Q} l'\ C(t)\ v$ are support functions for the convex closed bounded sets $\{B\ (t)\ P\}$ and $\{C\ (t)\ Q\}$. Consequently, the sets $\{B\ (t)\ U_1\}$ and $\{C\ (t)\ V_1\}$ of the vectors u° an v° on which the maximum on the right hand sides of (2.2) and (2.3) is achieved when $l = l_0$ are the subdifferentials of the corresponding support functions at point l_0 [4]. Since function $\varkappa\ (l,\ t)$ is convex in l and is [2.3] the support function of set $H\ (t)$ of (2.5), its subdifferential $H_1\ (t)$ at point $l = l_0$ in sum with $\{C\ (t)\ V_1\}$ yields the set $\{B\ (t)\ U_1\}$. Hence follows the validity of inclusion (2.5). It remains to show that functions $u^\circ\ (t,\ l_0)$ and $v^\circ\ (t,\ l_0)$ can be chosen measurable. Indeed, the sets $\{B\ (t)\ U_1\}$, $\{C\ (t)\ V_1\}$ and $H_1\ (t)$ are upper semi-continuous by inclusion as t varies; therefore, we can choose [1, 5] measurable functions $C\ (t)\ v^\circ\ (t,\ l_0) \in \{C\ (t)\ V_1\}$ and $h^\circ\ (t,\ l_0) \in H_1\ (t)$, and, then, $B\ (t)\ u^\circ\ (t,\ l_0)$ being the sum of two measurable functions, is measurable too.

Let us now define the first player's strategy $\mathbf{U}^*$. Suppose that some position $\{t,\ x\ [t]\}$ has been realized. On the interval $t \leqslant \tau \leqslant \vartheta$ we choose controls $u^\circ\ (\cdot\ l_0) = u^\circ\ (\tau,\ l_0)$ and $v^\circ\ (\cdot\ l_0) = v^\circ(\tau,\ l_0)$ which satisfy the maximum conditions (2.3) and (2.4) for almost all $\tau \in [t, \vartheta]$ and for which inclusion (2.5) holds. We consider the motion $x\ (\tau;\ t,\ x\ [t],\ u^\circ\ (\cdot\ l_0),\ v^\circ\ (\cdot\ l_0))$, $\tau \in [t,\ \vartheta]$ of system (1.1), generated by the controls $u = u^\circ\ (\cdot\ l_0)$ and $v = v^\circ\ (\cdot\ l_0)$ under the initial condition $x\ (t;\ t,\ x\ [t],\ u^\circ\ (\cdot\ l_0),\ v^\circ\ (\cdot\ l_0)) = x\ [t]$.

D e f i n i t i o n 1. Let an m-dimensional vector $s\ (t)$ be defined by the equality

$$s\ (t) = -\ x\ (\vartheta;\ t,\ x\ [t],\ u^\circ\ (\cdot\ l_0),\ v^\circ\ (\cdot\ l_0)) \tag{2.6}$$

Then the first player's strategy $\mathbf{U}^*$ is defined in the following manner:

1) if $s\ (t)$ is a nonzero vector for a position $\{t,\ x\ [t]\}$ then with this position we associate a set $U^*\ (t,\ x\ [t])$ of all vectors u^* which satisfy the maximum

condition

$$s'(t)B(t)u^* = \max_{u\in P} s'(t)\ B(t)\ u \tag{2.7}$$

2) if, however, $s(t)$ is a zero vector for a position $\{t,\ x[t]\}$, then we assume that $U^*(t,\ x[t]) = P$.

From the Cauchy formula determining $x(\vartheta;\ t,\ x[t],\ u^\circ(\cdot l_0),\ v^\circ(\cdot l_0))$ and from the results in [1] it follows that strategy $\mathbf{U}^*$ defined by conditions 1) and 2) is admissible.

Theorem 1. If sets P and Q are convex and Conditions A is fulfilled, then the first player's strategy $\mathbf{U}^*$ constructed in accord with Definition 1), guarantees him the game result $(\gamma[\vartheta]\,|\,t_0,\ x_0,\ \mathbf{U}^*,\ v) \leqslant \varepsilon_0(t_0,\ x_0)$ under any admissible control method of the second player.

Proof. Consider the function

$$\varepsilon[t] = \varepsilon(t,\ x[t]) = \|\ x(\vartheta;\ t,\ x[t],\ u^\circ(\cdot l_0),\ v^\circ(\cdot l_0))\ \|^2$$

Strategy $\mathbf{U}^*$ is admissible and, therefore, the derivative $d\varepsilon[t]\ /\ dt$ defined by

$$d\varepsilon[t]/dt = 2s'(t)\ [h^\circ(t,\ l_0) - \{B(t)\ u[t] - C(t)\ v[t]\}]$$

exists for almost all t. By the construction of set $H(t)$ for any admissible realization $v[t]$ we can find an admissible control $u^{(1)}(t)$ for which

$$h^\circ(t,\ l_0) = \{B(t)\ u^{(1)}(t) - C(t)\ v[t]\}$$

Therefore,

$$d\varepsilon[t]/dt = 2s'(t)\ \{B(t)\ u^{(1)}(t) - B(t)\ u[t]\}$$

From this equality and maximum condition (2.7) it follows that when $u[t] = u^*$ the inequality $d\varepsilon[t]\ /\ dt \leqslant 0$ is valid for almost all t for any position $\{t,\ x\}$ at which $\varepsilon[t] > 0$. Now taking into account that the equalities $\varepsilon[t_0] = \varepsilon_0^2(t_0,\ x_0)$ and $\gamma^2[\vartheta] = \varepsilon[\vartheta]$ hold by the definition of the auxiliary function $\varepsilon[t]$, we conclude that the theorem's assertion is valid.

The second player's strategy $\mathbf{V}^*$, solving Problem 2, is constructed similarly. Let the function $\varkappa(l,\ t)$ of (2.1), appearing in Condition A, be concave in l for each $t \in [t_0,\ \vartheta]$; then by analyzing the set

$$G(t) = \bigcap_{u\in P} \{B(t)\,u - C(t)\,Q\}$$

instead of set $H(t)$, we can prove a lemma similar to Lemma 1. The second player's strategy $\mathbf{V}^*$ is specified by the set $V^*(t,\ x[t])$ of vectors v^* satisfying the maximum condition

$$s'(t)\,C(t)\,v^* = \max_{v\in Q} s'(t)\,C(t)\,v$$

at positions $\{t, x[t]\}$ for which $\| s(t) \| \neq 0$, while $V^*(t, x[t]) = Q$ at positions for which $s(t) = 0$. The next statement can be proved by the same plan as the proof of Theorem 1.

Theorem 2. If sets P and Q are convex and the function $\varkappa(l, t)$ of (2.1) is concave in l for each $t \in [t_0, \vartheta]$, then the second player's strategy $\mathbf{V}^*$ guarantees him the game result $(\gamma[\vartheta] \mid t_0, x_0, u, \mathbf{V}^*) \geqslant \varepsilon_0(t_0, x_0)$ under any admissible control method of the first player.

Notes. 1°. Condition A can be weakened. As the proof of Theorem 1 shows, to construct the strategy $\mathbf{U}^*$ solving Problem 1 it is sufficient that for the initial position $\{t_0, x_0\}$ there exist optimal program controls $u^\circ(t, l_0)$ and $v^\circ(t, l_0)$, $t_0 \leqslant t \leqslant \vartheta$, satisfying maximum conditions (2.3) and (2.4), for which the inclusion

$$\{B(t) P\} \supset \{C(t) Q\} + h^\circ(t, l_0)$$

is fulfilled for almost all $t \in [t_0, \vartheta]$. In this case the assumption on the convexity of sets P and Q is unessential and can be dropped.

2°. A singularity of the control method proposed, in comparison with the extremal aiming rule developed in [1], is that the vector $s(t)$ used in the definitions of the player's strategies is generally easier to compute than the corresponding vector $l^\circ[t] = l^\circ(t, x[t])$ in the extremal construction. This is due to the fact that to determine the vector $l^\circ[t]$ it is necessary to solve the extremal problem (1.3) for each current position $\{t, x[t]\}$. Whereas to compute the vector $s(t)$ of (2.6) we need to know the solution of problem (1.3) only for the initial position $\{t_0, x_0\}$. It is clear that the result obtained is worse than when using the extremal aiming rule [1] because not all of the opponent's "errors" are taken advantage of. It should be noted that in comparison with the direct methods in game theory [6] and with the *a priori* stable paths [2] the control method we have proposed is more complicated but yields a better result from the view-point of one of the players. Thus, the method described above for solving Problems 1 and 2 falls inbetween the extremal aiming rule and the direct methods in differential game theory.

3°. It can be varified that the control procedure suggested for the first player takes system (1.1) into the position $\{x\} = 0$ no later than at the program absorption instant $\vartheta_0(t_0, x_0)$ under any admissible realization $v[t]$, $t_0 \leqslant t \leqslant \vartheta_0$ of the second player's control.

3. As an example we consider a guidance problem for a conflict-controlled material point moving along a horizontal straight line. The point's equations of motion are

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = u - v; \quad |u| \leqslant \mu, \quad |v| \leqslant \nu, \quad \mu > \nu. \tag{3.1}$$

Let the game's payoff γ estimate the distance of the phase point $x[\vartheta]$ at a specified instant ϑ from the origin $x_1 = x_2 = 0$, i.e.,

$$\gamma[\vartheta] = \{x_1^2[\vartheta] + x_2^2[\vartheta]\}^{1/2}$$

All the hypotheses of Theorem 1 are fulfilled for system (3.1); therefore, the first player's strategy U^* can be constructed as in Definition 1. As in [1], we select the following initial data $x_{01} = -7$, $x_{02} = 4$, $t_0 = 0$, $\vartheta = 4$, $\mu = 2$, and $\nu = 1$. Having made the necesssary computations, we get that $\varepsilon_0(t_0, x_0) = 1$, the maximum on the right hand side of (1.3) is achieved on the vector $l_0 = (-1, 0)$ and the vector $s(t)$ of (2.6) is determined by the equalities

$$s_1(t) = -x_1[t] - x_2[t](\vartheta - t) + {}^1/_2(\vartheta - t)^2$$
$$s_2(t) = -x_2[t] + \vartheta - t$$

The first player's strategy U^* is determined as follows:

1) If $s_1(t)(\vartheta - t) + s_2(t) \neq 0$ for a position $\{t, x_1[t], x_2[t]\}$ then the set $U^*(t, x_1[t], x_2[t])$ consists of the single point

$$u^*[t] = 2\,\mathrm{sign}\,\{s_1(t)(\vartheta - t) + s_2(t)\}$$

2) If $s_1(t)(\vartheta - t) + s_2(t) = 0$ for a position $\{t, x_1[t], x_2[t]\}$, then $U^*(t, x_1[t], x_2[t]) = P$, i.e., $u^*[t]$ is an arbitrary quantity satisfying the inequality $-2 \leqslant u^*[t] \leqslant 2$; to be specific we assume that $u^*[t] = 0$ in this case.

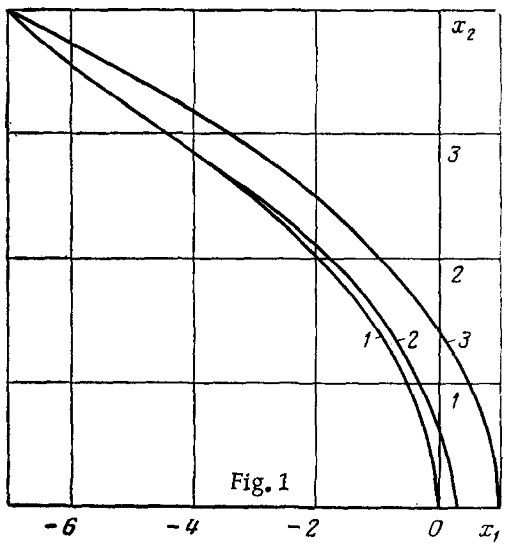

Fig. 1

The realizations of the motions dictated by the different choices of strategies of the first and second players were calculated on a computer and are shown in Fig. 1. Curve 1 shows the phase trajectory generated by the first player's optimal extremal strategy U^c [1], under the condition that the second player selects the control $v \equiv 0$. Curve 2 shows the phase trajectory corresponding to the first player's strategy U^* described in the present article, when the second player's control is $v \equiv 0$. As expected, we see that the magnitude $\gamma[\vartheta] = 0$ is realized in the first case, while a large value of payoff $\gamma[\vartheta]$, equal to 0.258, is realized in the second case. Curve 3 is generated by the pair $\{U^\circ, V^\circ\}$ of optimal extremal strategies [1]; the motion corresponding to the strategy pair $\{U^*, V^\circ\}$ takes place along this same curve. We note, further, that the a p r i o r i stable path [2] constructed for this example also lies along curve 3.

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